Levitation of a Rapidly Oscillating Magnetic Dipole Above a Metallic Sheet

B Shayak

Department of Theoretical and Applied Mechanics,
Sibley School of Mechanical and Aerospace Engineering,
Cornell University,
Ithaca – 14853,
New York State, USA

sb2344@cornell.edu , shayak.2015@iitkalumni.org

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Abstract

We show that such a magnetic dipole suspended at a height h above a conducting sheet experiences a lift force proportional to $1/h^2$. This represents an order of magnitude improvement over the $1/h^4$ lift obtained in the quasistatic limit.

Introduction

There are several ways of achieving stable spatial confinement of a magnetic levitator [1-7]. One of them is to use eddy currents or induced currents [8,9]. Being dynamical, they naturally fall outside the limitations of Earnshaw's theorem [10,11]. A significant result in this field has been demonstrated by Saslow [14]. He has used Maxwell's method of retreating images to determine the electromagnetic fields and hence the lift force on an oscillating dipole suspended vertically above a metallic sheet. It is shown that when the oscillation frequency becomes higher than the inverse of the diffusion time constant of the metal, the sheet acts like a Type I superconductor in that it expels the magnetic field completely from inside it. In this regime, the metal sheet can be replaced by an image dipole, which exerts a lift force proportional to the inverse fourth power of the height of the dipole above the sheet. Since changing the position of the source automatically changes the position of the image, this arrangement does not have a lateral instability coming as 'baggage' with the vertical stability. Two other approaches to magnetic levitation can be found in References [15] and [16] – the former uses a rotating saddle trap while the latter uses hydrodynamic stabilization to counteract Earnshaw's theorem.

In the present Letter, we consider the situation of Reference [14] but in a different limit. In [14], the calculations are in the quasistatic limit, where the $1/c^2\ \partial\mathbf{E}/\partial t$ term in $\nabla\times\mathbf{B}$ is neglected. However, it is a fact that the high-frequency response of dynamical systems is sometimes considerably different from the low-frequency one – an example is the Kapitza pendulum [17] where stabilization of the inverted position occurs at high frequency alone. To this end, we explore here the effects of applying a very high oscillation frequency on the dipole. As we shall see later, rapid oscillation means that the wavelength of the emitted waves is smaller than the height at which the dipole is suspended – slow oscillation on the other hand denotes the quasistatic limit. We find that in the high-frequency limit, the lift force becomes more long-ranged, varying as the inverse square of the height as against the inverse fourth.

In the process of calculation we also introduce a technique for obtaining electromagnetic radiation fields which we believe is new. This approach involves directly solving the wave equation for the electric and magnetic fields instead of first calculating the retarded potentials. The advantage of this procedure is that it can readily yield solutions for extended charge and current configurations, which are typically encountered in practice. Whereas the exact solution of radiation fields for point sources using retarded potentials is well-known [18], solving for extended sources is a different matter. Abbott and Griffiths [19] have calculated the fields outside a solenoid using an approximate retarded potential. A subtlety of the calculation has been highlighted by McDonald [20] who has first differentiated the retarded potential to find the fields and only then imposed the approximation. A perturbative approach to the problem has been adopted by Roy et. al. [21]. The method we propose here however cannot be found in any of these or in the standard texts [22-24] on electromagnetism. We will first show how this method can be used to obtain the electromagnetic fields inside and outside a current-carrying spherical shell of finite radius R. After presenting the general solution, we take the limit $R \rightarrow 0$ for further calculation i.e. we shrink the current source down to a point dipole. We note that in this limit the fields could also have been obtained from [18].

Calculation of the electromagnetic fields of a macroscopic dipole

We consider a spherical shell of radius R centred at the origin which carries a surface current $\mathbf{K} = K_0 \sin\theta \cos\omega t \hat{\boldsymbol{\varphi}}$ where ω is high enough so that the shell radiates instead of generating a quasistatic field. We choose this configuration because it corresponds to a single angular harmonic of the wave equation. Moreover, it is well-known that in the absence of the time dependence, this current distribution generates outside the sphere a magnetic field entirely equivalent to that of a point dipole $m\hat{\mathbf{z}}$ with $m=4\pi K_0 R^3/3$ placed at its centre. Note that at this stage we do not treat R to be vanishingly small and K_0 correspondingly large.

Maxwell's equations for this configuration can be combined into the wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \omega K_0 \delta(r-R) \sin\theta \sin\omega t \hat{\boldsymbol{\varphi}} \quad , \tag{1}$$

where $\mu_0 \varepsilon_0 = 1/c^2$. The electric field satisfies the homogeneous equation everywhere except at the surface of the shell; we expect that the shell will provide boundary conditions which match the internal and external solutions. Since the source term is entirely in $\hat{\boldsymbol{\varphi}}$ direction, the net $\mathbf{E}$ too will be in that direction, although the equation (1) itself will hold only for Cartesian components of $\mathbf{E}$. In spherical polar coordinates r,θ,φ the homogeneous wave equation with the relevant temporal and angular dependences has the following solution set :

$$E(\mathbf{r},t)=\{j_1(kr),y_1(kr)\}\sin\theta\{\cos\varphi,\sin\varphi\}\{\cos\omega t,\sin\omega t\}\ , \qquad (2)$$

where E denotes any one Cartesian component of electric field, $k=\omega/c$ and every possible combination of functions separated by commas makes up a solution. Here the spherical Bessel functions are defined as

$$j_1(kr)=\frac{\cos kr}{kr}-\frac{\sin kr}{k^2r^2}\quad , \qquad (3a)$$

$$y_1(kr)=\frac{\sin kr}{kr}+\frac{\cos kr}{k^2r^2}\quad , \qquad (3b)$$

(choice of signs and normalization is to be noted).

The boundary conditions on **E** and **B** at the shell are that **E** is continuous, the normal component of **B** is also continuous and the tangential component of **B** has a jump proportional to the surface current. **E** has two components, a cosωt and a sinωt both of which have the spatial dependences indicated in (2). Since the current is cosωt, the jump in **B** comes only from its cosωt component; since **E** depends on $\partial\mathbf{B}/\partial t$, the jump will come from the sinωt component of **E**. So we focus on this component first.

Since $j_1(kr)$ is tractable at r=0 while $y_1(kr)$ is not, we try the ansatz

$$\mathbf{E}_{1,\sin}=A_1 j_1(kr)\sin\theta\sin\omega t\hat{\boldsymbol{\varphi}}\quad , \qquad (4a)$$

$$\mathbf{E}_{2,\sin}=\left(A_2 j_1(kr)+B_2 y_1(kr)\right)\sin\theta\sin\omega t\hat{\boldsymbol{\varphi}}\quad , \qquad (4b)$$

inside and outside the shell respectively. Here the subscripts 1 and 2 denote the electric fields inside and outside the shell while subscripts ‘sin’ and ‘cos’ refer to the temporal behaviour. Continuity of **E** at the shell $r=R$ gives the equation

$$A_1 j_1(kR)=A_2 j_1(kR)+B_2 y_1(kR)\quad , \qquad (5)$$

and we have one relation connecting A_1, A_2 and B_2.

Now we account for the jump in **B**. Its component tangent to the shell is B_θ; $\nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t$ yields

$$-\frac{\partial B_{\theta 1}}{\partial t}=-A_1\left(\frac{j_1(kr)}{r}+kj_1{}'(kr)\right)\sin\theta\sin\omega t\quad , \qquad (6)$$

where prime denotes differentiation with respect to kr. This implies that

$$B_{\theta 1}=-\frac{A_1}{\omega}\left(\frac{j_1(kr)}{r}+kj_1{}'(kr)\right)\sin\theta\cos\omega t\quad . \qquad (7)$$

Similarly calculating B_θ outside the shell, we can write the boundary condition

$$\left[B_{\theta 2}-B_{\theta 1}\right]_{r=R}=\mu_0 K_0 \sin\theta\cos\omega t \quad , \tag{8}$$

as

$$\begin{aligned}&A_1\left(\frac{j_1(kR)}{R}+kj_1'(kR)\right)-A_2\left(\frac{j_1(kR)}{R}+kj_1'(kR)\right)\\&-B_2\left(\frac{y_1(kR)}{R}+ky_1'(kR)\right)=\mu_0 K_0\omega\end{aligned} \quad , \tag{9}$$

and we have a second relation between the three unknown constants.

Continuity of the radial component of magnetic field fails to yield the third relation. For we have

$$B_{r1}=\frac{2A_1 j_1(kr)}{\omega r}\cos\theta\cos\omega t \quad , \tag{10a}$$

$$B_{r2}=\left(\frac{2A_2 j_1(kr)}{\omega r}+\frac{2B_2 y_1(kr)}{\omega r}\right)\cos\theta\cos\omega t \quad , \tag{10b}$$

and equating the two at $r=R$ leads to the condition (5) again.

We now bring the $\cos\omega t$ component of **E** into focus. Like (4), this must be having the form

$$\mathbf{E}_{1,\cos}=C_1 j_1(kr)\sin\theta\cos\omega t\hat{\boldsymbol{\varphi}} \quad , \tag{11a}$$

$$\mathbf{E}_{2,\cos}=\left(C_2 j_1(kr)+D_2 y_1(kr)\right)\sin\theta\cos\omega t\hat{\boldsymbol{\varphi}} \quad , \tag{11b}$$

inside and outside. **E** is continuous at the shell so

$$C_1 j_1(kR)=C_2 j_1(kR)+D_2 y_1(kR) \quad . \tag{12}$$

The $\cos\omega t$ component of **E** generates a $\sin\omega t$ component of **B**; since the current does not have this component, B_θ is continuous also. The boundary condition is the same as (9) with zero in place of $\mu_0 K_0\omega$ and we have

$$\begin{aligned}&C_1\left(\frac{j_1(kR)}{R}+kj_1'(kR)\right)-C_2\left(\frac{j_1(kR)}{R}+kj_1'(kR)\right)\\&-D_2\left(\frac{y_1(kR)}{R}+ky_1'(kR)\right)=0\end{aligned} \quad . \tag{13}$$

Finally, the boundary condition is imposed that there can only be outgoing waves outside the sphere. At large distances the dominant term is $1/r$; at this order (4b) yields $A_2 \cos kr \sin\omega t + B_2 \sin kr \sin\omega t$, while (11b) yields $C_2 \cos kr \cos\omega t + D_2 \sin kr \cos\omega t$. To express them as trigonometric functions of $kr-\omega t$ we must have $D_2=-A_2$ and $C_2=B_2$. Before writing out the resulting system, we define $F(kR)$ as

$j_1(kR)/R + kj_1'(kR)$ and $G(kR)$ as $y_1(kR)/R + ky_1'(kR)$, to achieve notational compactness. Then we have

$$f(kR)A_1 - f(kR)A_2 - g(kR)B_2 = 0 \quad , \tag{14a}$$

$$F(kR)A_1 - F(kR)A_2 - G(kR)B_2 = \mu_0 K_0 \omega \quad , \tag{14b}$$

$$g(kR)A_2 - f(kR)B_2 + f(kR)C_1 = 0 \quad , \tag{14c}$$

$$G(kR)A_2 - F(kR)B_2 + F(kR)C_1 = 0 \quad , \tag{14d}$$

$$C_2 = B_2 \quad , \tag{14e}$$

$$D_2 = -A_2 \quad . \tag{14f}$$

We can solve this linear system to determine the unknown constants and hence **E** and **B** everywhere. It is noteworthy that this approach yielded the exact solution for the macroscopic configuration and bypassed the retarded potential altogether.

Calculation of the levitation force

In this Section, to simplify the solution of (14) and achieve a better comparison with existing literature, we shrink the macroscopic dipole to a point dipole i.e. we take the limit $K_0 \rightarrow \infty$ and $R \rightarrow 0$ while keeping $K_0 R_0^3 = 3m/4\pi$ a constant. At really small kR, the asymptotic forms $j_1(kR) = -kR/3$ and $y_1(kR) = 1/k^2R^2$ are relevant, and (14) becomes

$$-\frac{k}{3}A_1 + \frac{k}{3}A_2 - \frac{1}{k^2R^3}B_2 = 0 \quad , \tag{15a}$$

$$-\frac{2k}{3}A_1 + \frac{2k}{3}A_2 + \frac{1}{k^2R^3}B_2 = \frac{\mu_0 \omega m}{4\pi R^3} \quad , \tag{15b}$$

$$\frac{1}{k^2R^2}A_2 + \frac{kR}{3}B_2 - \frac{kR}{3}C_1 = 0 \quad , \tag{15c}$$

$$\frac{1}{k^2R^3}A_2 - \frac{2k}{3}B_2 + \frac{2k}{3}C_1 = 0 \quad . \tag{15d}$$

This system is solved to obtain $B_2 = \mu_0 m \omega k^2/4\pi$ and $A_2 = 0$. The resulting solution agrees with what has been found in [18].

We now use the above reckoning to obtain the force on the dipole suspended at a height h above the conducting sheet $z=0$. It is a well known fact that a time-dependent electric field penetrates a conductor to a distance of the order of the skin depth, which varies as $\omega^{-1/2}$ [22]. So long as h exceeds the skin depth by more than an order of magnitude, we can treat the conductor to be a perfect flux-expeller. In this case, we have the boundary condition that **E** and **B** are zero at z=0. We have **E** outside the dipole as

$$\mathbf{E} = \frac{\mu_0 m\omega k^2}{4\pi}\left(\frac{\cos(kr-\omega t)}{kr} - \frac{\sin(kr-\omega t)}{k^2 r^2}\right)\sin\theta\hat{\boldsymbol{\varphi}} \quad . \tag{16}$$

We expect that the condition of zero **E** at z=0 will be satisfied by an image configuration which replaces the flux-expeller with a dipole at (0,0,–h); the following Figure determines the sign of the image. The electric field far away from the dipole is given by (16). Considering the source and image and the point (x,y,0) on the plane, at any instant of time, the r and t parts of **E** are the same for both the dipoles. As for the sinθ component, we can see that θ for the source is $\pi/2+\beta$ while θ for the image is $\pi/2-\beta$ for some angle β between 0 and $\pi/2$.

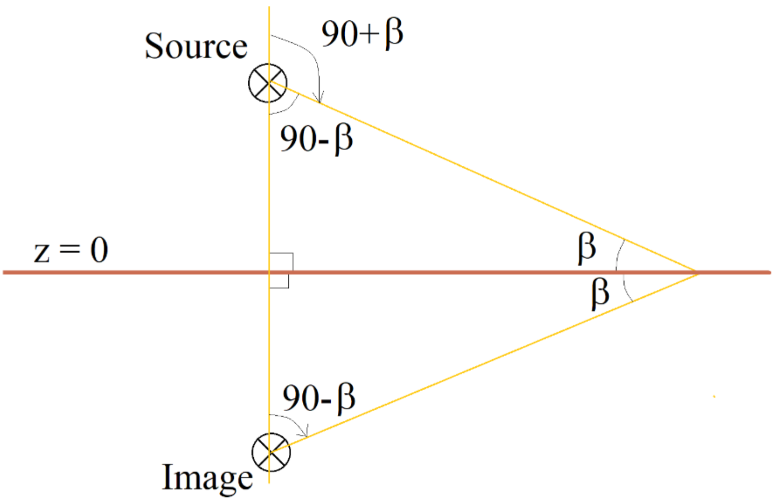

Figure 1 : Schematic diagram showing the source and image dipoles with respect to the sheet.

Since sin($\pi/2+\beta$) = sin($\pi/2-\beta$) for $0 \le \beta \le \pi/2$, this factor is also identical for both dipoles. Thus, the image must be the negative of the source i.e. if the source dipole is $m\cos\omega t\hat{\mathbf{z}}$, the image is $-m\cos\omega t\hat{\mathbf{z}}$.

To find the force on the source, we use the standard formula $\mathbf{F} = (m\cdot\nabla)\mathbf{B}$. The source dipole is in the z-direction only. Further, due to the symmetry of the problem, we expect that the force itself will also be in z-direction only. Hence this formula reduces to $F = m(\partial B_z/\partial z)$. B_z at the source point comes from both source and image – the source can't be exerting a force on itself, so the entire contribution will be that of the image. Thus, we must find B_z at the point (0,0,h) on account of the image $-m\cos\omega t$ at (0,0,$-h$).

Evidently, for the image point, $\hat{\mathbf{z}} = \hat{\mathbf{r}}$ and B_z=B_r. Thus, we should use the formula (10b), after substituting the B_2 we found later and replacing m by $-m$. This gives

$$B_r = -\frac{\mu_0 m}{2\pi}\left(\frac{k\sin kr}{r^2} + \frac{\cos kr}{r^3}\right)\cos\theta\cos\omega t \quad . \tag{17}$$

Although (17) does not represent an outgoing wave, the $\sin\omega t$ component of B_r will not generate a force with non-zero time average, hence we neglect it. For the configuration under study, θ=0; now replacing r by z gives

$$B_z = -\frac{\mu_0 m}{2\pi}\left(\frac{k\sin kz}{z^2} + \frac{\cos kz}{z^3}\right)\cos\omega t \quad . \tag{18}$$

Its $\partial/\partial z$ is

$$\frac{\partial B_z}{\partial z} = -\frac{\mu_0 m}{2\pi}\left(\frac{k^2\cos kz}{z^2} - \frac{3k\sin kz}{z^3} - \frac{3\cos kz}{z^4}\right)\cos\omega t \quad . \tag{19}$$

For the force we want $m\,(\partial Bz/\partial z)$ averaged over time – since m is also $\cos\omega t$, the average evaluates to 1/2. Then we substitute $z = 2h$ and get

$$F = -\frac{\mu_0 m^2}{4\pi}\left(\frac{k^2\cos 2kh}{4h^2} - \frac{3k\sin 2kh}{8h^3} - \frac{3\cos 2kh}{16h^4}\right) \quad , \tag{20}$$

which is our final expression.

Discussion and conclusion

We close with a brief discussion of our results. The force (20) contains three terms where decreasing powers of k are compensated by increasing powers of $1/h$. When k is very small compared to $1/h$, the dominant term is the third one. This is the result which Saslow [14] has obtained – it is eminently sensible since $k \rightarrow 0$ means that the oscillation frequency of the dipole goes to zero i.e. we approach the quasistatic limit. This is the regime in which Saslow's calculation has been carried out. On the other hand, if k is large compared to $1/h$, the dominant term in the force (20) is the first one. This force is positive i.e. a lift force when $\cos 2kh$ is negative. It has a much longer-ranged dependence on h than the force in the quasistatic case. Large k corresponds to high frequency of oscillation of the dipole. Here, the comparison scale is $1/h$. If h equals 1 cm then large k implies waves of wavelength 1 mm or smaller i.e. k greater than or equal to approximately 150 m^{-1}. This corresponds to ω greater than or equal to approximately $4.5\text{x}10^{10}$ s^{-1}. These frequencies are smaller than the inverse of the time constants involved in the dynamics of electrons in the metal (10^{-14} s) [21] so the assumption of the metallic sheet as a flux expeller does not break down. At the frequencies involved, the skin depth in the metal is $2/(\mu_0\sigma\omega)^{1/2}$ which, for copper, evaluates to approximately 1 micrometre. So the step of replacing the metal by an ideal flux-expeller is valid.

The long-ranged dependence of the force on r opens up interesting practical possibilities for levitating objects magnetically. In slow limit, the $1/h^4$ dependence means that dipoles located somewhat far away from each other do not affect each other at all. With $1/h^2$ however, it should be possible to manufacture setups where dipoles interact with further away ones to augment the lift and confinement forces. We also note that the ω involved will have to be such that the typical wavelength of the waves is of the order of the gap between the dipole and the sheet, i.e. about 1 mm. This occurs at microwave frequencies. We leave the design and analysis of magnetic levitators based on this principle to future study.

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